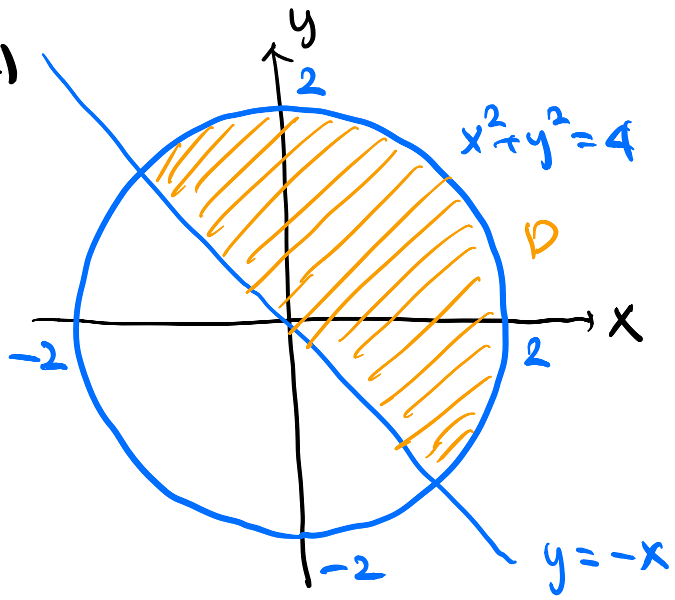
1. [10 points] Consider the function $f(x, y) = x - y$ on the region D in $\mathbb{R}^2$ defined by $x + y \geq 0$ and $x^2 + y^2 \leq 4$.

a. [5 points] Sketch the region D.

b. [5 points] Calculate the integral of f over D.

(a)



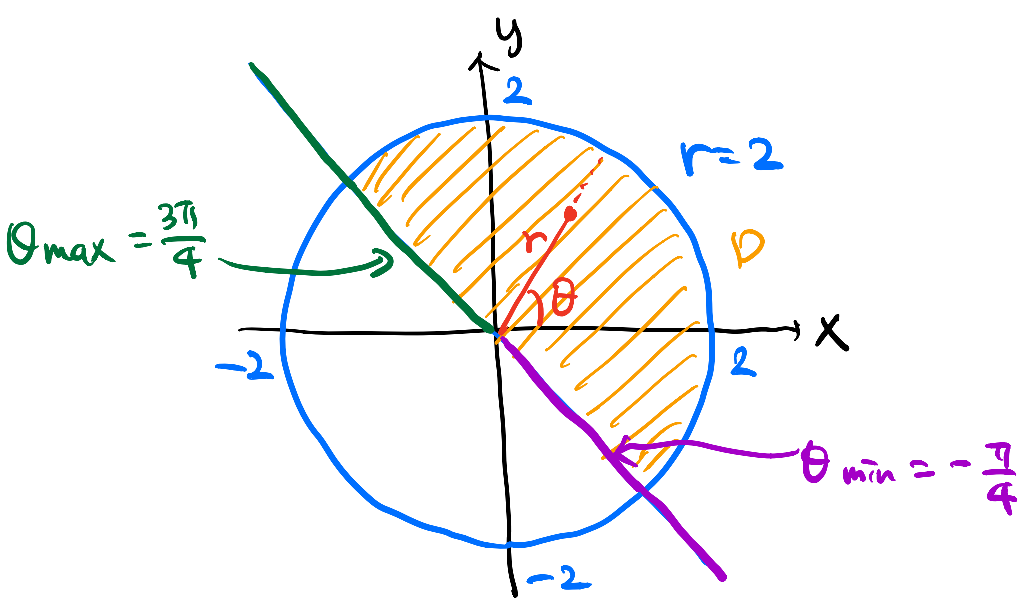
$x+y \geq 0$: above the line

$x+y=0$ or $y=-x$

$x^2+y^2 \leq 4$: inside the circle

$x^2+y^2=4$.

(b) <u>Sol 1</u> (Polar coordinates)



$\theta_{min} = -\frac{\pi}{4}$

$\theta_{max} = \frac{3\pi}{4}$

$0 \leq r \leq 2$

Jacobian

$$\iint_D x - y \, dA = \int_{-\pi/4}^{3\pi/4} \int_0^2 (r\cos\theta - r\sin\theta) \cdot r \, dr \, d\theta$$

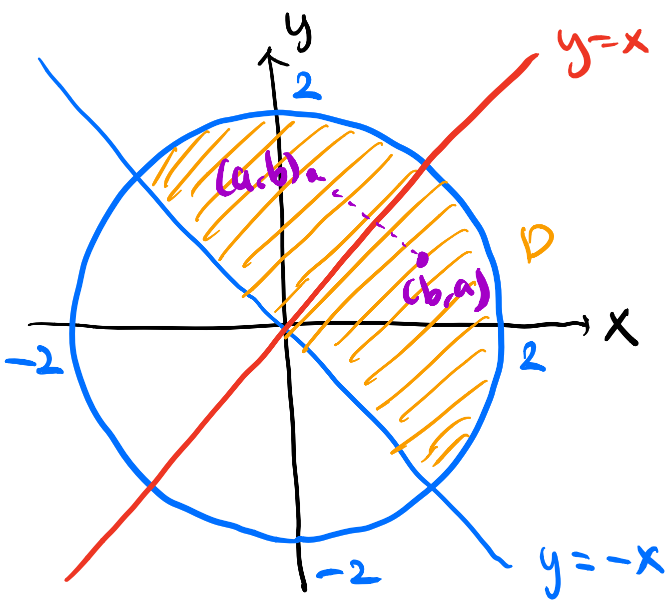
$$= \int_{-\pi/4}^{3\pi/4} \int_0^2 r^2\cos\theta - r^2\sin\theta \, dr \, d\theta$$

$$= \int_{-\pi/4}^{3\pi/4} \frac{r^3}{3} (\cos\theta - \sin\theta) \Big|_0^2 \, d\theta$$

$$= \int_{-\pi/4}^{3\pi/4} \frac{8}{3} (\cos\theta - \sin\theta) \, d\theta$$

$$= \frac{8}{3} (\sin\theta + \cos\theta) \Big|_{-\pi/4}^{3\pi/4} = \boxed{0}$$

Sol 2 (Symmetry)



y = x : line of symmetry for D.

A point (a,b) reflects to (b,a) along this line of symmetry.

$f(a,b) = a - b$, $\quad f(b,a) = b - a \Rightarrow f(a,b) = -f(b,a)$

These values "cancel out" in $\iint_D f(x,y)\,dA$.

$\Rightarrow \iint_D f(x,y)\,dA = 0$.

2. [12 points] Find and classify the critical points for the function $h(x, y) = x^4 + y^3 - 6y - 2x^2$.

$h_x = 4x^3 - 4x$, $h_y = 3y^2 - 6$.

$\nabla h = (4x^3 - 4x, 3y^2 - 6)$

$\nabla h = \vec{0} \Rightarrow 4x^3 - 4x = 0$ and $3y^2 - 6 = 0$.

$4x^3 - 4x = 0 \rightsquigarrow x(x^2 - 1) = 0 \rightsquigarrow x = 0, \pm 1$.

$3y^2 - 6 = 0 \rightsquigarrow y^2 = 2 \rightsquigarrow y = \pm\sqrt{2}$.

Critical points : $(-1, \pm\sqrt{2})$, $(0, \pm\sqrt{2})$, $(1, \pm\sqrt{2})$

Use the 2^{nd} derivative test.

$$h_{xx} = \frac{\partial h_x}{\partial x} = \frac{\partial}{\partial x}(4x^3 - 4x) = 12x^2 - 4.$$

$$h_{xy} = \frac{\partial h_x}{\partial y} = \frac{\partial}{\partial y}(4x^3 - 4x) = 0.$$

$$h_{yy} = \frac{\partial h_y}{\partial y} = \frac{\partial}{\partial y}(3y^2 - 6) = 6y.$$

Hessian $D = h_{xx} \cdot h_{yy} - h_{xy}^2 = ((12x^2 - 4) \cdot 6y - 0^2$

$= 24(3x^2 - 1)y$.

At $(-1,\sqrt{2})$: $D = 24\left(3(-1)^2-1\right)\sqrt{2} > 0$.

$h_{xx} = 12\cdot(-1)^2 - 4 > 0$

$\rightsquigarrow \boxed{\text{loc. min at } (-1,\sqrt{2})}$

At $(-1,-\sqrt{2})$: $D = 24\cdot(3(-1)^2-1)(-\sqrt{2}) < 0$.

$\rightsquigarrow \boxed{\text{saddle pt at } (-1,-\sqrt{2})}$

At $(0,\sqrt{2})$: $D = 24\cdot(3\cdot 0^2-1)\cdot\sqrt{2} < 0$

$\rightsquigarrow \boxed{\text{saddle pt at } (0,\sqrt{2})}$

At $(0,-\sqrt{2})$: $D = 24\cdot(3\cdot 0^2-1)(-\sqrt{2}) > 0$

$h_{xx} = 12\cdot 0^2 - 3 < 0$

$\rightsquigarrow \boxed{\text{loc. max at } (0,-\sqrt{2})}$

At $(1,\sqrt{2})$: $D = 24\cdot(3\cdot 1^2-1)\cdot\sqrt{2} > 0$

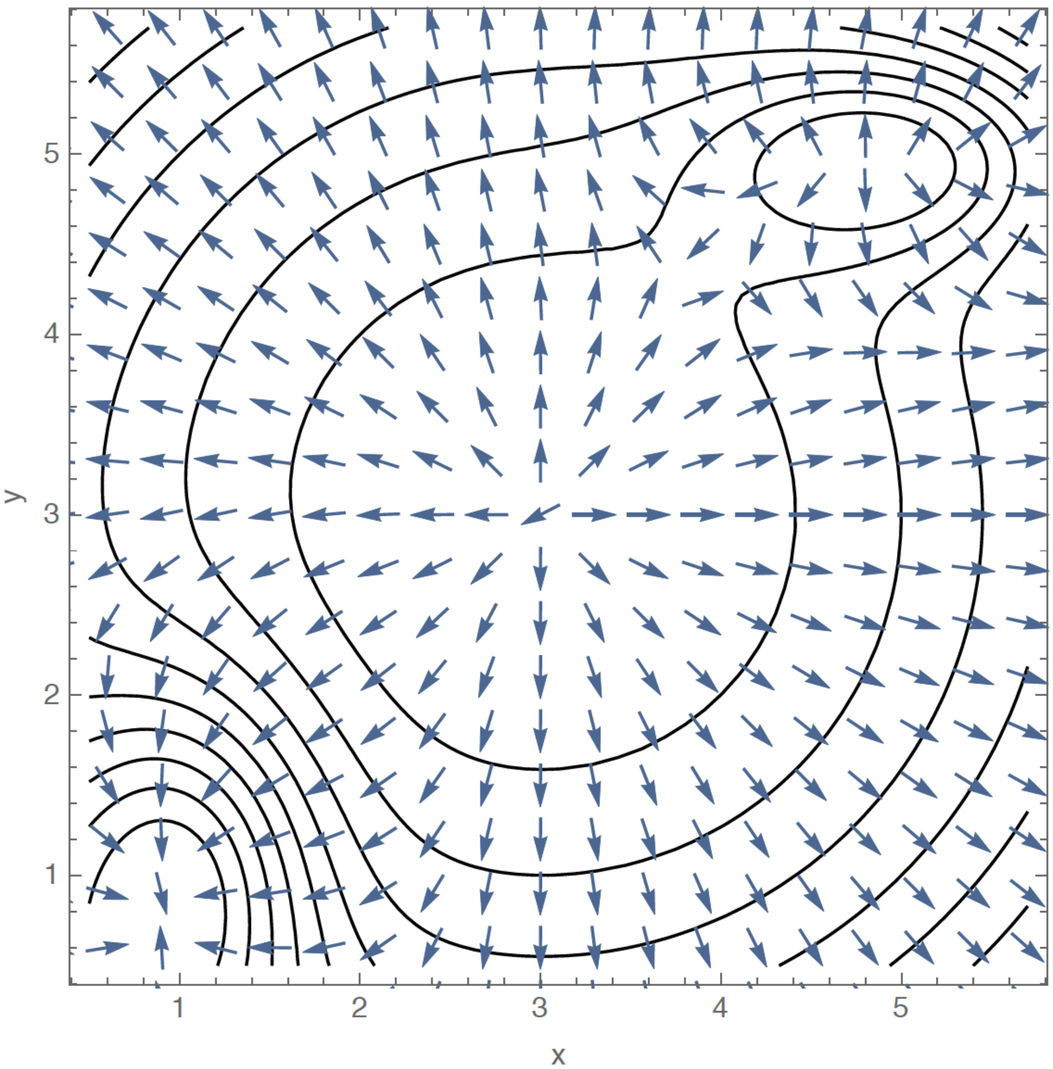
$h_{xx} = 12\cdot 1^2 - 3 > 0$

$\rightsquigarrow \boxed{\text{loc. min at } (1,\sqrt{2})}$

At $(1,-\sqrt{2})$: $D = 24(3\cdot 1^2-1)(-\sqrt{2}) < 0$.

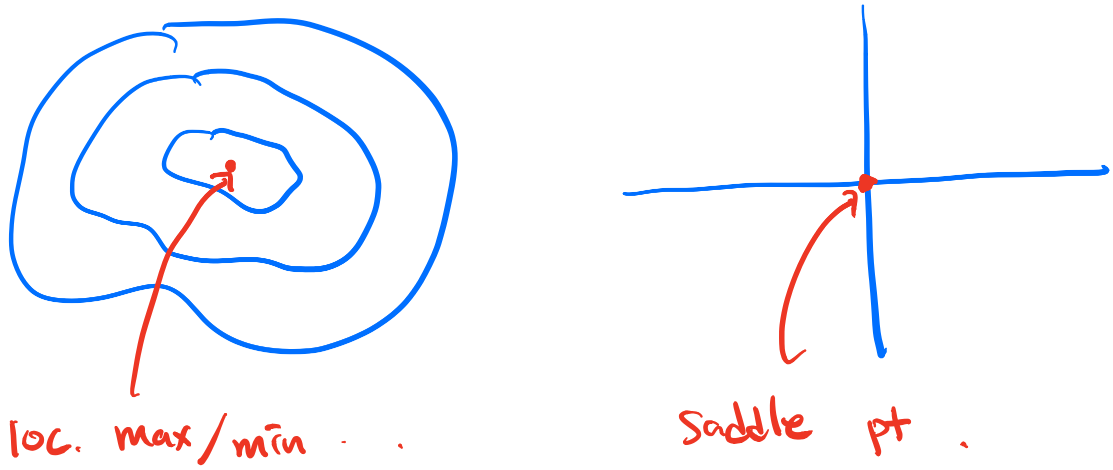
$\rightsquigarrow \boxed{\text{saddle pt at } (1,-\sqrt{2})}$

3. [15 points] The graph below is a plot of some of the level curves of a function g in a rectangular region $R = [.4, 5.8] \times [.4, 5.8]$. Assume that as we move between adjacent level curves the value of g increases or decreases by exactly one. The arrows point in the direction of ∇g.

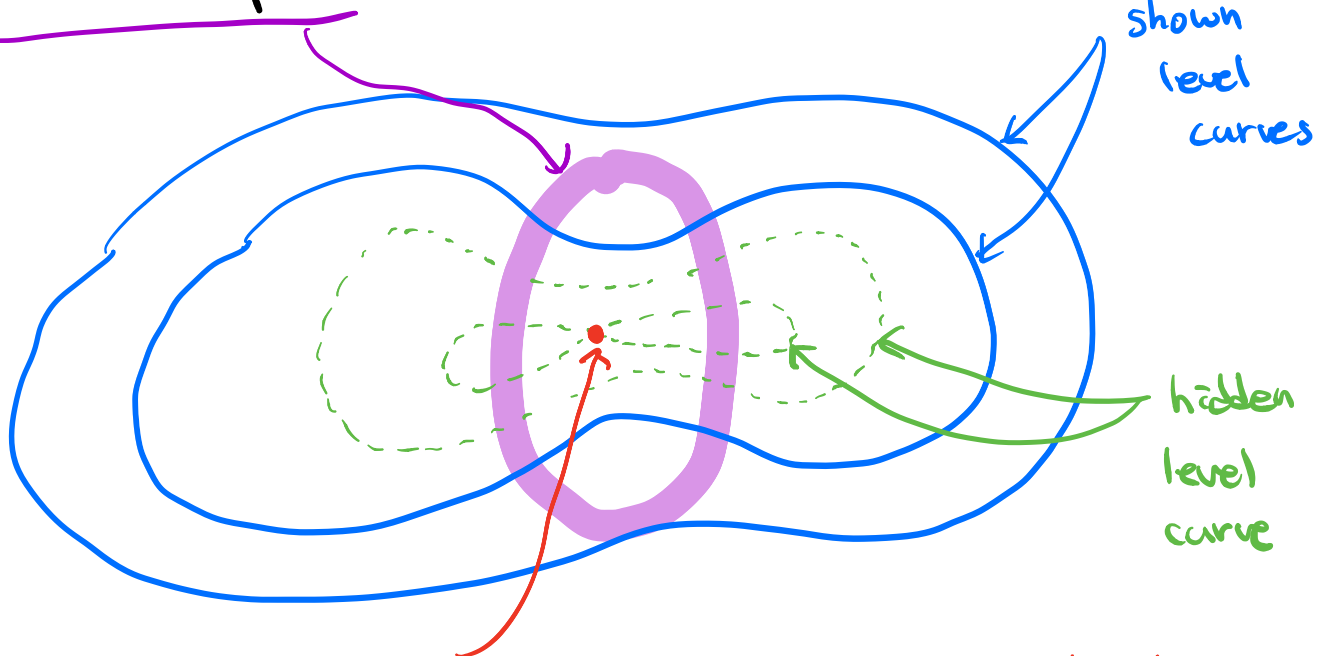


a. [9 points] Identify the approximate coordinates[1] of the critical points of g in R. For each critical point, indicate if it is a point where the function has a local maximum, a local minimum, or a saddle.

Recall: crit. pts typically look as follows:



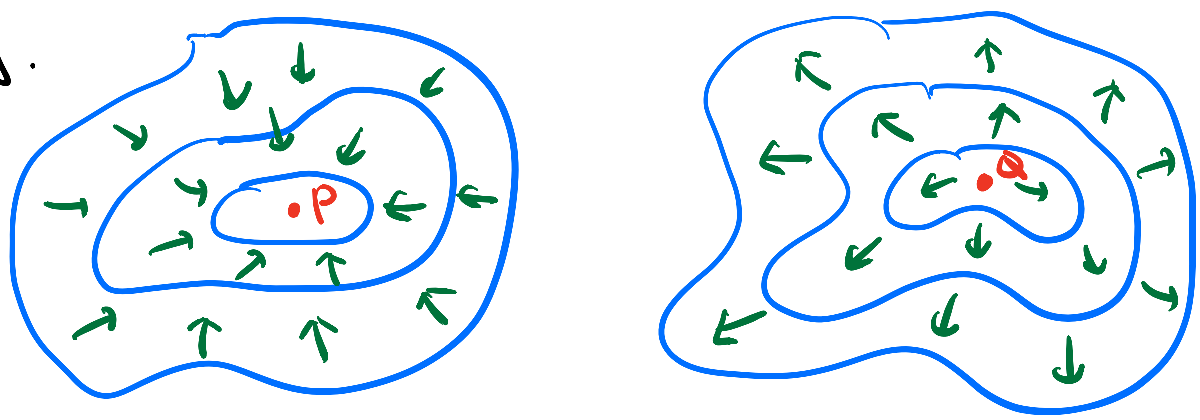
For saddle pts, you can also look for "peanut shapes"



There will be an intersection of hidden level curves $\rightsquigarrow$ (likely) a saddle pt.

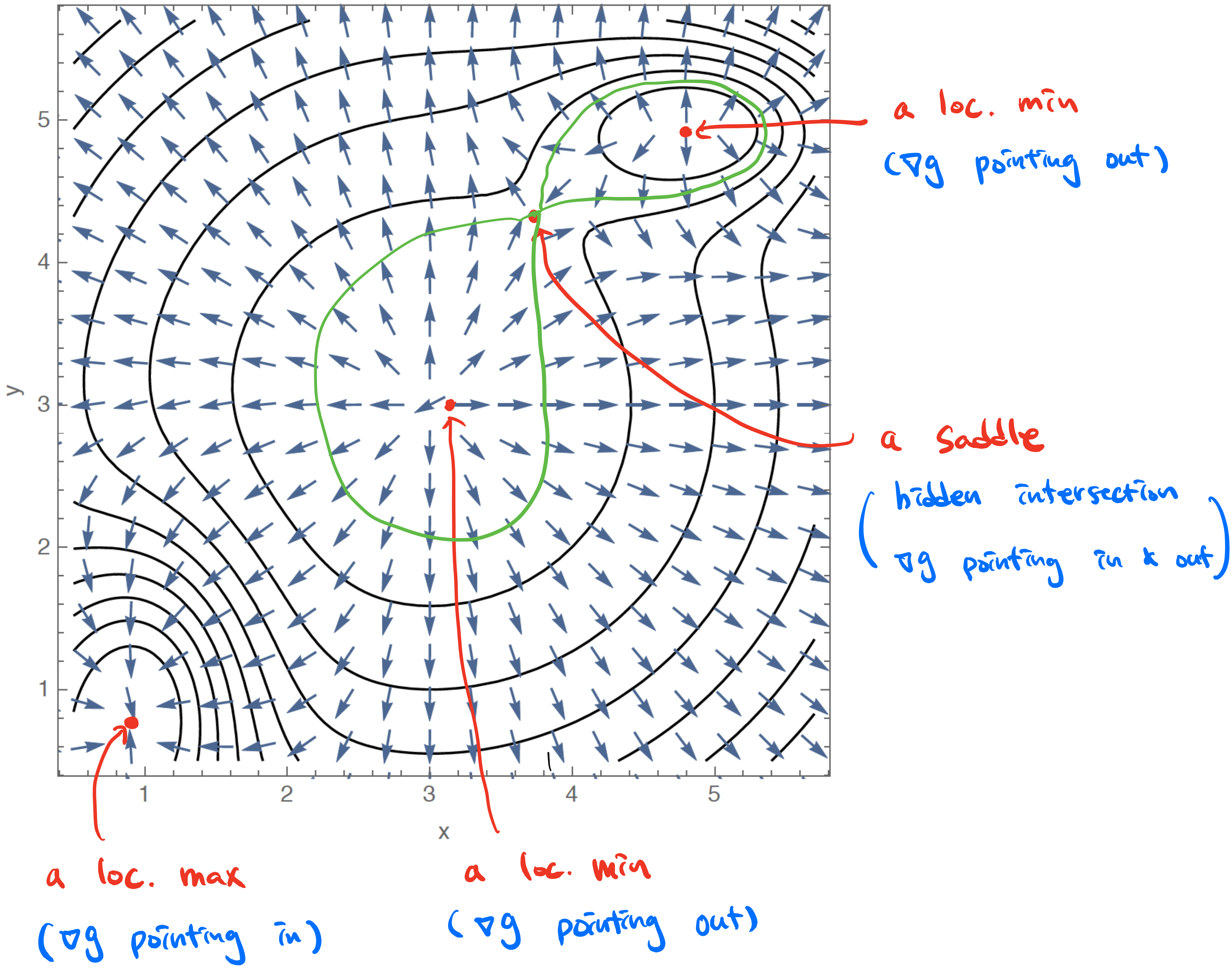
For local max/mins, look at arrows to figure out whether it's a max or a min ($\because \nabla g$ points in direction of increase)

e.g.



$\rightsquigarrow$ g increasing toward P, decreasing toward Q

$\Rightarrow$ P is a loc. max, Q is a loc. min.



$$\Rightarrow \boxed{\begin{array}{lcl} (0.9, 0.8) & : & \text{a loc. max} \\ (3.2, 3) & : & \text{a loc. min} \\ (3.8, 4.3) & : & \text{a saddle pt} \\ (4.8, 4.9) & : & \text{a loc. min.} \end{array}}$$

b. [4 points] Identify the approximate coordinates[2] of the points where the function g attains its global maximum and global minimum over the rectangle R.

R : closed and bounded

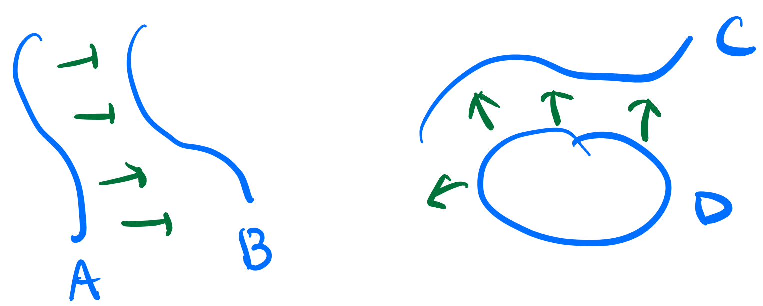
$\rightsquigarrow$ need to check crit. pts and boundary.

Note: consecutive levels differ by 1.

& $\underbrace{\text{arrows}}_{=\nabla g}$ point in direction of increase

$\Rightarrow$ We can find relative positions of all levels

e.g.



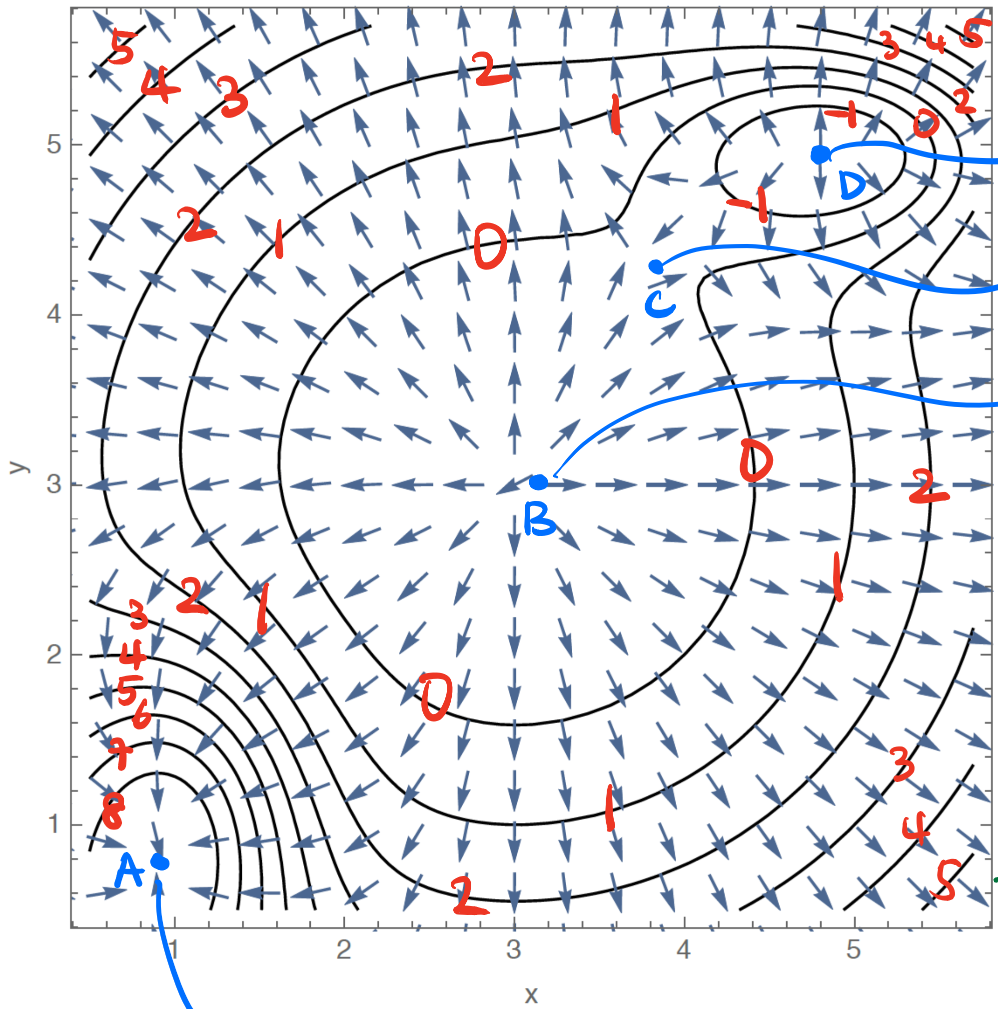
Level B = Level $A+1$, Level C = Level $D+1$.

Idea: Set one curve to be at level 0,

and find levels for all other curves.

Then compare values at crit. pts

and boundary.



Value ≈ -1.5

Value ≈ -0.3

Value ≈ -0.5

* $B < C$ because arrows point from B to C.

boundary value range: $2.5 \sim 8.2$

Value ≈ 8.5

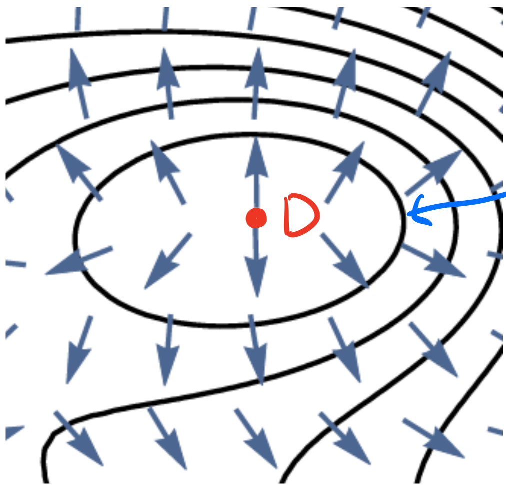
* Value at A is bigger than boundary because arrows point to A from boundary

$\Rightarrow$ A > boundary > C > B > D.

$\Rightarrow$ $\boxed{\begin{array}{l} \text{global max at } A = (0.9, 0.8) \\ \text{global min at } D = (4.8, 4.9) \end{array}}$

c. [2 points] If the value of g at its global minimum on R is between 23 and 24, then the value of the global maximum of g is between _______ and _______.

global min at (4.8, 4.9) and value between 23 and 24.

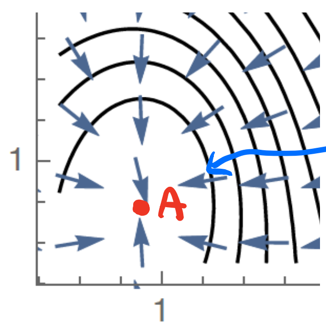


This curve must be at level 24.

(It was set at −1 in the picture in (b))

So in our picture from (b), levels must be shifted by 25.

global max at (0.9, 0.8):

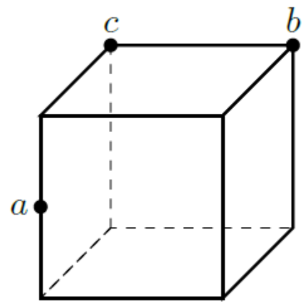


This curve must be at level 8 + 25 = 33.

⇒ Value at A must be between

$\boxed{\text{33 and 34.}}$

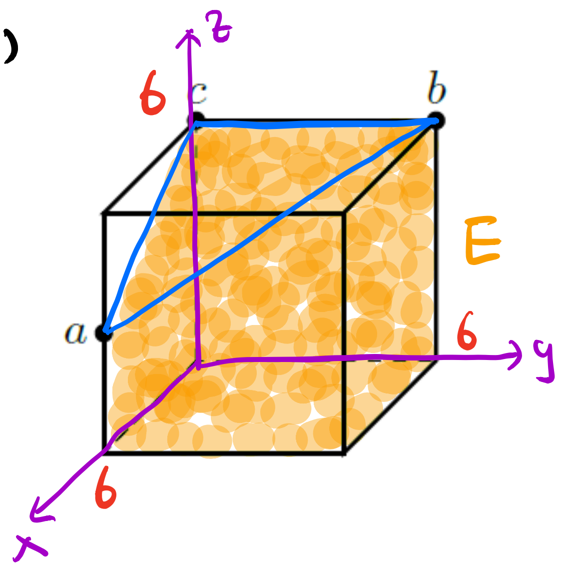
4. [10 points] The sides of the cube below have length six. The point a is at the midpoint of its edge. Let P be the plane that contains the points a, b, and c.



a. [8 points] Set up, but do not evaluate, an integral for the volume of that part of the cube that lies below the plane P.

b. [2 points] Calculate the volume of that part of the cube that lies below the plane P.

(a)



$a : (6, 0, 3)$

$b : (0, 6, 6)$

$c : (0, 0, 6)$

$(6, 0, -3)$
$(0, 6, 0)$

Find the equation of the plane.

$\overrightarrow{ca} = (6, 0, -3)$, $\overrightarrow{cb} = (0, 6, 0)$

$\rightsquigarrow$ normal vector $\vec{n} = \overrightarrow{ca} \times \overrightarrow{cb} = (18, 0, 36)$

The plane contains $c = (0, 0, 6)$

$\rightsquigarrow 18(x-0) + 0 \cdot (y-0) + 36(z-6) = 0$

$\rightsquigarrow x + 2(z-6) = 0$

$\rightsquigarrow z = 6 - \frac{x}{2}$.

We look at the solid under the graph $z = 6 - \frac{x}{2}$ and above the rectangular domain $R = [0,6] \times [0,6]$.

$$\Rightarrow \text{Volume} = \iint_R 6 - \frac{x}{2}\, dA$$

$$= \boxed{\int_0^6 \int_0^6 6 - \frac{x}{2}\, dx\,dy}$$

* Note: Can also use the dydx order.

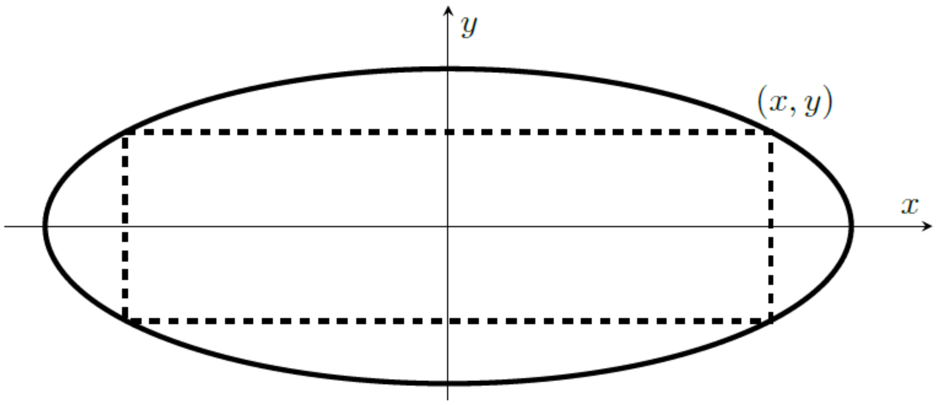
(b) Volume $= \int_0^6 \int_0^6 6 - \frac{x}{2}\, dx\,dy$

$$= \int_0^6 6x - \frac{x^2}{4} \Big|_0^6 \, dy$$

$$= \int_0^6 27 \cdot dy$$

$$= 27 \cdot 6 = \boxed{162}$$

5. [12 points] Suppose a, b are positive constants.
In this problem we will consider rectangles that are inscribed in the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The sides of the inscribed rectangles are parallel to the coordinate axes, as in the figure below.



a. [10 points] Using the method of Lagrange multipliers, find the rectangle of largest area that can be inscribed in the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. What is its area?
No credit given for solutions that do not use the method of Lagrange multipliers.

(x, y) : vertex on 1st quadrant

$\Rightarrow$ Area $= 2x \cdot 2y = 4xy$, $x > 0, y > 0$

$\rightsquigarrow$ Maximize $f(x,y) = 4xy$ on $x, y > 0$

subject to $g(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$.

$\nabla f = (4y, 4x)$, $\nabla g = \left(\frac{2x}{a^2}, \frac{2y}{b^2} \right)$

We solve the system

$\nabla f = \lambda \nabla g$ and $g = 0$

$$\Rightarrow \begin{cases} (4y, 4x) = \lambda \left(\frac{2x}{a^2}, \frac{2y}{b^2} \right) & \text{—— (1)} \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 & \text{—— (2)} \end{cases}$$

$$(1):\ \begin{cases} 4y = \dfrac{2\lambda x}{a^2} \rightsquigarrow y = \dfrac{\lambda x}{2a^2} \quad (*) \\[2ex] 4x = \dfrac{2\lambda y}{b^2} \rightsquigarrow 4x = \dfrac{\lambda^2 x / a^2}{b^2} = \dfrac{\lambda^2 x}{a^2 b^2}. \end{cases}$$

$$\underset{x\neq 0}{\Rightarrow}\quad 4 = \frac{\lambda^2}{a^2 b^2} \Rightarrow \lambda = \pm 2ab.$$

$$(*):\ y = \frac{\lambda x}{2a^2} = \pm \frac{2abx}{2a^2} = \pm \frac{b}{a} x.$$

$$\text{But } x, y, a, b > 0 \rightsquigarrow y = \frac{b}{a} x. \quad (**)$$

$$\text{Constraint}:\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{a^2} + \frac{b^2 x^2 / a^2}{b^2} = 1$$

$$\Rightarrow \frac{2x^2}{a^2} = 1 \Rightarrow x = \frac{a}{\sqrt{2}} \quad (\because x > 0)$$

$$(**):\ y = \frac{b}{a} x = \frac{b}{a} \cdot \frac{a}{\sqrt{2}} = \frac{b}{\sqrt{2}}$$

$$\Rightarrow (x, y) = \left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right) \text{ is the only solution.}$$

$$f\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right) = 4 \cdot \frac{a}{\sqrt{2}} \cdot \frac{b}{\sqrt{2}} = 2ab$$

$$\Rightarrow \boxed{\text{Max area of } 2ab \text{ at } \left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)}$$

6. [15 points] Suppose $g\colon \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, and consider the iterated double integral

$$\int_0^4 \int_{y^{1/2}}^2 g(x,y)\,dx\,dy.$$

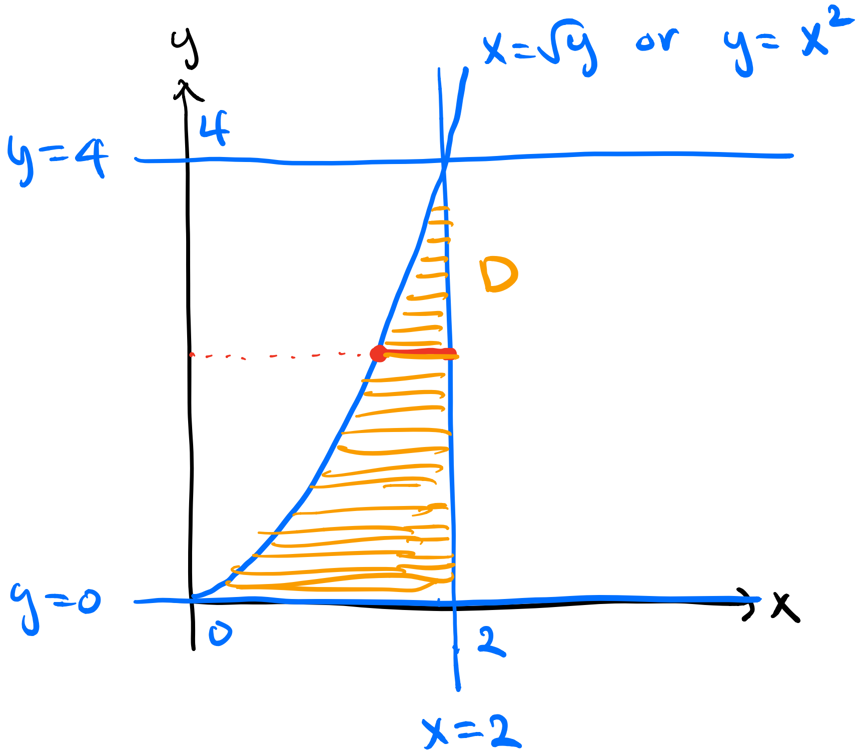
In this integral, x is innermost and y is outermost.

a. [5 points] Sketch the region of integration.

b. [5 points] Rewrite the integral with y innermost and x outermost.

c. [5 points] Evaluate the integral for $g(x,y) = \sin(x^3 - 1)$.

(a) Domain D : $\underbrace{0 \le y \le 4}_{\text{bounds for outer integral}}$, $\underbrace{\sqrt{y} \le x \le 2}_{\text{bounds for inner integral}}$.



(b) Describe D in the opposite order :

$$D = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 2,\ 0 \le y \le x^2\}$$

$$\Rightarrow \boxed{\int_0^2 \int_0^{x^2} g(x,y)\,dy\,dx}$$

(c) Use the integral from part (b).

$$\int_0^2 \int_0^{x^2} \underbrace{\sin(x^3-1)\,dy\,dx}_{\text{constant integral}}$$

$$= \int_0^2 x^2 \sin(x^3-1)\,dx$$

$$(u = x^3-1 \Rightarrow du = 3x^2\,dx)$$

$$= \int_{-1}^{7} \frac{1}{3} \sin u \, du$$

$$= -\frac{1}{3} \cos u \Big|_{-1}^{7} = \boxed{\frac{1}{3}\left(\cos(-1) - \cos(7)\right)}$$

7. [15 points] Indicate if each of the following is true or false by circling the correct answer. No partial credit will be given.

a. [3 points] Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function and $(c,d) \in \mathbb{R}^2$. If $f_{xx}(c,d)f_{yy}(c,d) < [f_{xy}(c,d)]^2$ then f has a saddle point at (c,d).

The statement is $\boxed{\text{False}}$

This looks true, because of the second derivative test, but it is not. The issue is that the second derivative test applies only to critical points. So for this statement to be true, you need to assume that $\nabla f(c,d) = \vec{0}$.

b. [3 points] Suppose $h\colon \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function. Suppose $(a,b) \in \mathbb{R}^2$ and C is the curve in $\mathbb{R}^2$ described by the equation $h(x,y) = h(a,b)$. If ℓ is the tangent line to C at (a,b), then $\nabla h(a,b)$ is perpendicular to ℓ.

The statement is $\boxed{\text{True}}$.

This is in fact one of key properties of the gradient vector.

c. [3 points] If $f\colon [0,1] \to \mathbb{R}$ is continuous, then

$$\int_0^1 \int_0^1 f(t)f(s)\,ds\,dt = \left[\int_0^1 f(u)\,du\right]^2.$$

The statement is $\boxed{\text{True}}$

You can see this as follows:

$$\int_0^1 \int_0^1 f(t) f(s)\, ds\, dt = \int_0^1 f(t) \int_0^1 f(s)\, ds\, dt$$

$f(t)$ is constant for inner integral

$$= \int_0^1 f(s) ds \cdot \int_0^1 f(t) dt$$

$\int_0^1 f(s) ds$ is constant.

$$= \left(\int_0^1 f(u) du \right)^2$$

two integrals are the same.

d. [3 points] Suppose $g\colon \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function and $\mathbf{k} = \langle 0, 0, 1 \rangle$. We have

$$D_{\mathbf{k}} g(x, y, z) = g_z(x, y, z).$$

This statement is $\boxed{\text{True}}$.

To see this, you compute

$$D_{\vec{k}}\, g(x, y, z) = \nabla g(x, y, z) \cdot \vec{k}$$

$$= (g_x, g_y, g_z) \cdot (0, 0, 1)$$

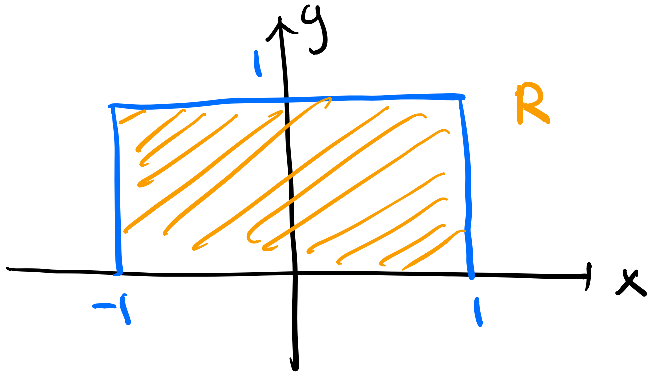
$$= g_z(x, y, z)$$

e. [3 points]

$$\int_{-1}^{1}\int_{0}^{1} e^{x^2+y^2}\sin(x)dy\,dx = 0.$$

The statement is $\boxed{\text{True}}$.

Note that the domain $R = [-1,1]\times[0,1]$ is symmetric about the y-axis.



Also, the integrand $f(x,y) = e^{x^2+y^2}\sin(x)$ is odd in x.

$\Rightarrow$ By symmetry, $\iint_R f(x,y)\,dA = 0$

* You can also see this as follows:

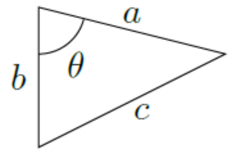
$$\int_{-1}^{1}\int_{0}^{1} e^{x^2+y^2}\sin(x)\,dy\,dx = \underbrace{\int_{0}^{1}\int_{-1}^{1} e^{x^2+y^2}\sin(x)\,dx\,dy}_{\text{Fubini}}$$

$$= \int_0^1 e^{y^2}\cdot \int_{-1}^{1} \underbrace{e^{x^2}\sin(x)}_{\text{odd}}\,dx\,dy$$

e^{y^2} is constant for dx

$$= \int_0^1 e^{y^2}\cdot 0\,dy = 0.$$

8. [11 points] Recall from your homework that the Law of Cosines states that for a triangle with sides of length a, b, and c we have $c^2 = a^2 + b^2 - 2ab\cos(\theta)$ where θ is the measure (in radians) of the angle opposite side c. Thus, the Law of Cosines implicitly defines θ as a function of the side lengths a, b, and c.



a. [3 points] Compute $\partial\theta/\partial c$.

Set $f(a,b,c,\theta) = a^2+b^2-c^2-2ab\cos\theta$.

The Law of Cosines : $f(a,b,c,\theta)=0$.

By the Implicit function theorem,

$$\frac{\partial\theta}{\partial c} = -\frac{f_c}{f_\theta} = -\frac{-2c}{2ab\sin\theta} = \boxed{\frac{c}{ab\sin\theta}}$$

b. [3 points] Compute $\partial\theta/\partial a$.

By the Implicit function theorem,

$$\frac{\partial\theta}{\partial a} = -\frac{f_a}{f_\theta} = -\frac{2a-2b\cos\theta}{2ab\sin\theta} = \boxed{\frac{b\cos\theta-a}{ab\sin\theta}}.$$

c. [5 points] Suppose the lengths of the sides of the triangle (measured in meters) are changing as a function of time (measured in seconds) according to the rules $a(t) = 3+t$, $b(t) = 3$, and $c(t) = 3+2t$. What is $d\theta/dt$ at time $t = 1$?

Consider θ as a function of a,b,c.

$$\Rightarrow \frac{d\theta}{dt} = \frac{\partial\theta}{\partial a}\cdot\frac{da}{dt} + \frac{\partial\theta}{\partial b}\cdot\frac{db}{dt} + \frac{\partial\theta}{\partial c}\cdot\frac{dc}{dt}$$

chain rule

$$\frac{da}{dt} = \frac{d}{dt}(3+t) = 1,$$

$$\frac{db}{dt} = \frac{d}{dt}(3) = 0.$$

$$\frac{dc}{dt} = \frac{d}{dt}(3+2t) = 2.$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{\partial\theta}{\partial a} + 2\frac{\partial\theta}{\partial c}$$

$$= \frac{b\cos\theta - a}{ab\sin\theta} + \frac{2c}{ab\sin\theta} \qquad (*)$$

(a) & (b)

At $t=1$, $a(1) = 4$, $b(1) = 3$, $c(1) = 5$.

$$\Rightarrow \cos\theta = \frac{a^2+b^2-c^2}{2ab} = \frac{4^2+3^2-5^2}{2\cdot 4\cdot 3} = 0$$

$$\Rightarrow \theta = \pi/2. \quad \Rightarrow \sin\theta = 1.$$

$$(*): \frac{d\theta}{dt} = \frac{3\cdot 0 - 4}{4\cdot 3\cdot 1} + \frac{2\cdot 5}{3\cdot 4\cdot 1} = \boxed{\frac{1}{2} \text{ (rad/sec)}}$$